MAGGIA LABORATORIO

Función característica

$$\varphi(t) = E\{\exp(\mathrm{i}t\mathsf{x})\}$$

$$\varphi(t) = \int_{-\infty}^{\infty} \exp(\mathrm{i}tx)\, f(x)\, \mathrm{d}x$$

$$\varphi(t) = \sum_i \exp(\mathrm{i}tx_i)\, P(\mathsf{x} = x_i)$$

$$\varphi^{(n)}(t) = \frac{\mathrm{d}^n\varphi(t)}{\mathrm{d}t^n} = \mathrm{i}^n \int_{-\infty}^{\infty} x^n \exp(\mathrm{i}tx)\, f(x)\, \mathrm{d}x$$

$$\varphi^{(n)}(0) = \mathrm{i}^n \lambda_n \quad .$$

$$\mathsf{y} = \mathsf{x} - \widehat{x} \longrightarrow \varphi_y(t) = \int_{-\infty}^{\infty} \exp\{\mathrm{i}\,t\,(x - \widehat{x})\}\, f(x)\, \mathrm{d}x = \varphi(t)\exp(-\mathrm{i}\,t\,\widehat{x})$$

Modos.....

$$\varphi_y^{(n)}(0) = \mathrm{i}^n \mu_n = \mathrm{i}^n E\{(\mathsf{x} - \widehat{x})^n\} \quad ,$$

$$\sigma^2(x) = -\varphi_y''(0) \quad .$$

Invirtiendo la función característica tenemos

$$\varphi(t) = \int_{-\infty}^{\infty} \exp(\mathrm{i}tx) f(x)\,\mathrm{d}x \quad \Longleftrightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\mathrm{i}tx)\varphi(t)\,\mathrm{d}t$$

Importancia: $\mathsf{w} = \mathsf{x} + \mathsf{y}$ Variables independientes

$$\varphi_w(t) = E[\exp\{\mathrm{i}t(\mathsf{x}+\mathsf{y})\}] = E\{\exp(\mathrm{i}t\mathsf{x})\exp(\mathrm{i}t\mathsf{y})\}$$

$$\varphi_w(t) = E\{\exp(\mathrm{i}tx)\}E\{\exp(\mathrm{i}ty)\} = \varphi_x(t)\varphi_y(t)$$

Ejemplo Poissoniana

$$\begin{aligned}\varphi(t) &= \sum_{k=0}^{\infty} \exp(itk)\frac{\lambda^k}{k!}\exp(-\lambda) = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{(\lambda\exp(it))^k}{k!} \\ &= \exp(-\lambda)\exp(\lambda e^{it}) = \exp\{\lambda(e^{it}-1)\} \quad .\end{aligned}$$

Sumamos dos poissonianas….

$$\begin{aligned} &= \exp\{\lambda_1(e^{it}-1)\}\exp\{\lambda_2(e^{it}-1)\} \\ &= \exp\{(\lambda_1+\lambda_2)(e^{it}-1)\} \quad .\end{aligned}$$

Función característica de una f.d.p. gaussiana, donde *a* es la media y b^2 es la varianza

$$\varphi(t) = \exp(ita)\exp(-\frac{1}{2}b^2t^2)$$

Reseña de la matemática involucrada en el concepto de función característica

Serie de Fourier

Tratamiento Tradicional en el análisis una serie temporal de datos

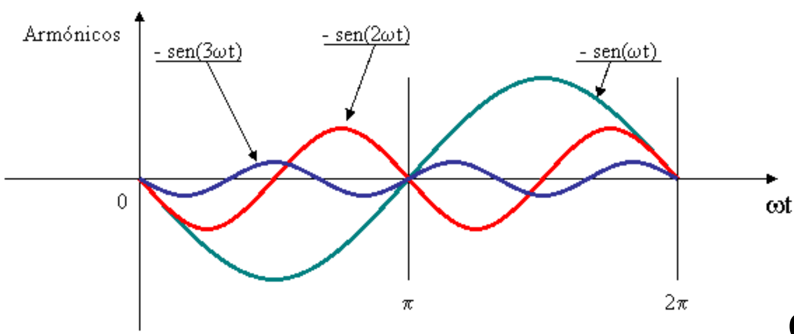
Representación matemática:

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \,\mathrm{sen}\left(\frac{2\pi n t}{T}\right) = a_0 + \sum_{n=1}^{\infty} c_n \exp\left(i\frac{2\pi n t}{T}\right)$$

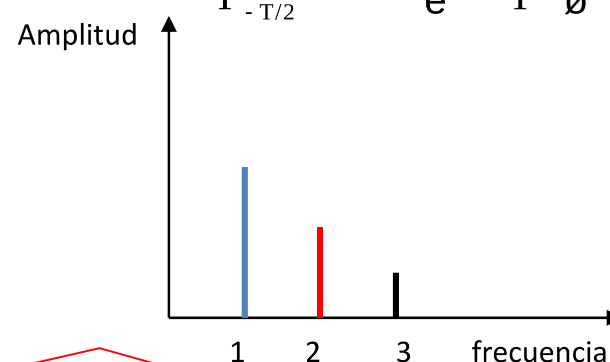
ω_0

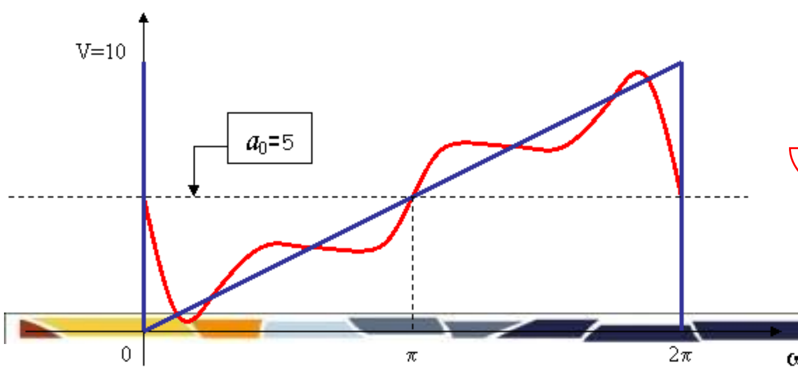
$$\left(\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{sen}\left(\frac{2\pi n t}{T}\right)dt \end{aligned}\right)$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp\left(-i\frac{2\pi n t}{T}\right)dt$$



$$c_n = \sqrt{a_n^2 + b_n^2}$$





Transformada de Fourier: para pasar del espacio del tiempo al de la frecuencia

$f(t)$

Transformada →

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(i\omega t)\,dt$$

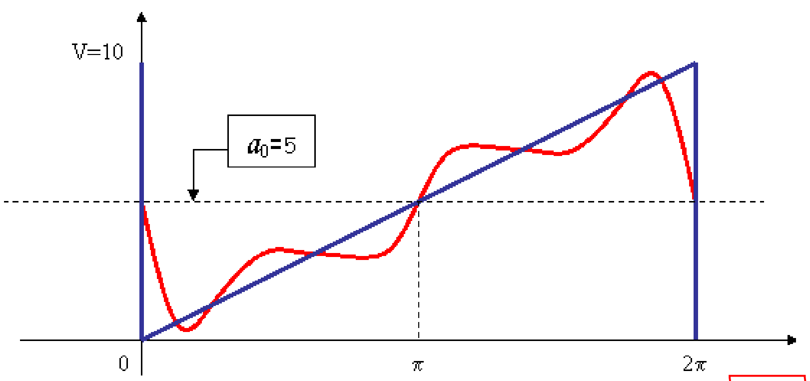
$$f(t) = \int_{-\infty}^{\infty} F(\omega)\exp(-i\omega t)\,d\omega$$

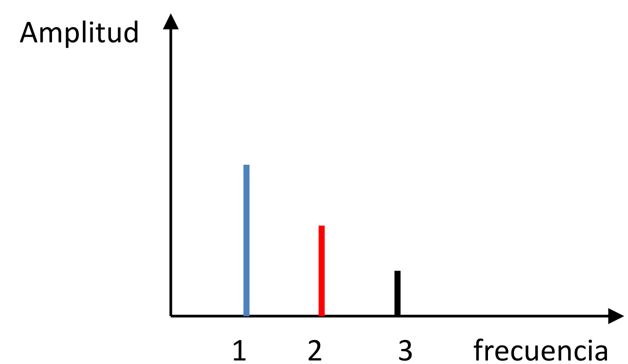
← Anti-transformada

Aspectos importantes:

$$F(\omega) = |F(\omega)|\exp(i\phi(\omega)) = \sqrt{a^2+b^2}\exp(i\phi(\omega))$$

$$= a(\omega) - ib(\omega)$$





Recordemos: $$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t}dt \quad \text{y} \quad T = \frac{2\pi}{\omega_0}$$

La serie de Fourier es:
-T/2< x < T/2

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t}dt\right]e^{in\omega_0 t}$$

O bien:

$$T = 2\pi/\omega_0$$

$$\omega_0 = 2\pi/T$$

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t}dt\right]\omega_0 e^{in\omega_0 t}$$

Cuando $T \to \infty$, $n\omega_0 \to \omega$ y $\omega_0 \to d\omega$ y el sumatorio se convierte en:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right]e^{i\omega t}d\omega$$

La transformada de Fourier

Es decir,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

← **Identidad de Fourier o antitrans-formada de Fourier**

donde:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

← **Transformada de Fourier**

Estas expresiones nos permiten calcular la expresión F(ω) (dominio de la frecuencia) a partir de f(t) (dominio del tiempo) y viceversa.

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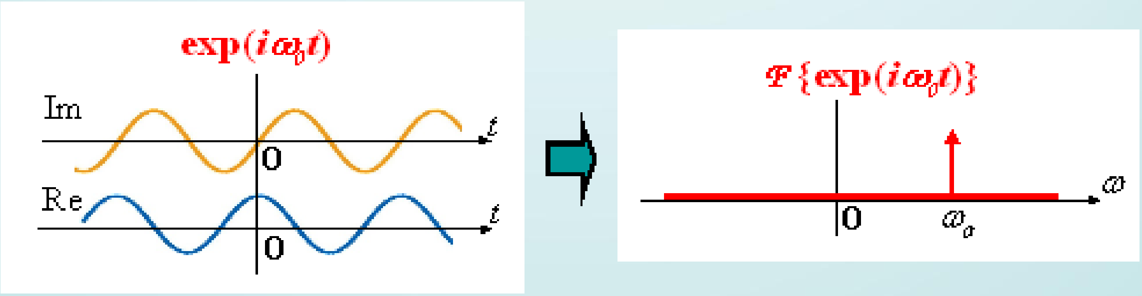
Ejemplos

La transformada de Fourier de la onda plana $\exp(i\omega_0 t)$

$$F\{e^{i\omega_0 t}\} = \int_{-\infty}^{\infty} e^{i\omega_0 t}\, e^{-i\omega t}\, dt =$$

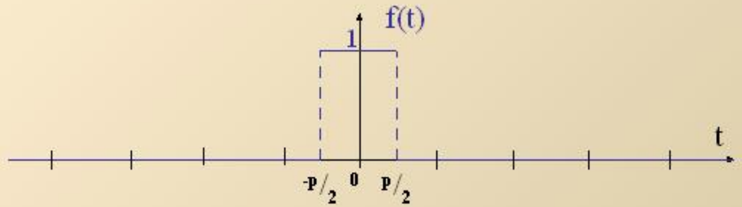
$$\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t}\, dt = 2\pi\,\delta(\omega-\omega_0)$$



La TF de $\exp(i\omega_0 t)$ es una frecuencia pura.

Ejemplos

Ejemplo. Calcular F(ω) para el pulso rectangular f(t) siguiente:



Solución. La expresión en el dominio del tiempo de la función es:

$$f(t)=\begin{cases}0 & t<\frac{-p}{2}\\ 1 & \frac{-p}{2}<t<\frac{p}{2}\\ 0 & \frac{p}{2}<t\end{cases}$$

Integrando:

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt=\int_{-p/2}^{p/2} e^{-i\omega t}dt$$

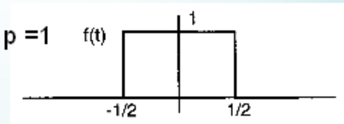
$$=\frac{1}{-i\omega}e^{-i\omega t}\Big|_{-p/2}^{p/2}=\frac{1}{-i\omega}(e^{-i\omega p/2}-e^{i\omega p/2})$$

Usando la fórmula de Euler:

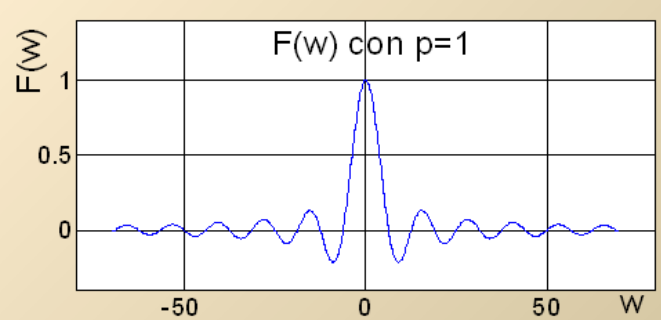
$$sen(\omega p/2)=\frac{e^{i\omega p/2}-e^{-i\omega p/2}}{2i}$$

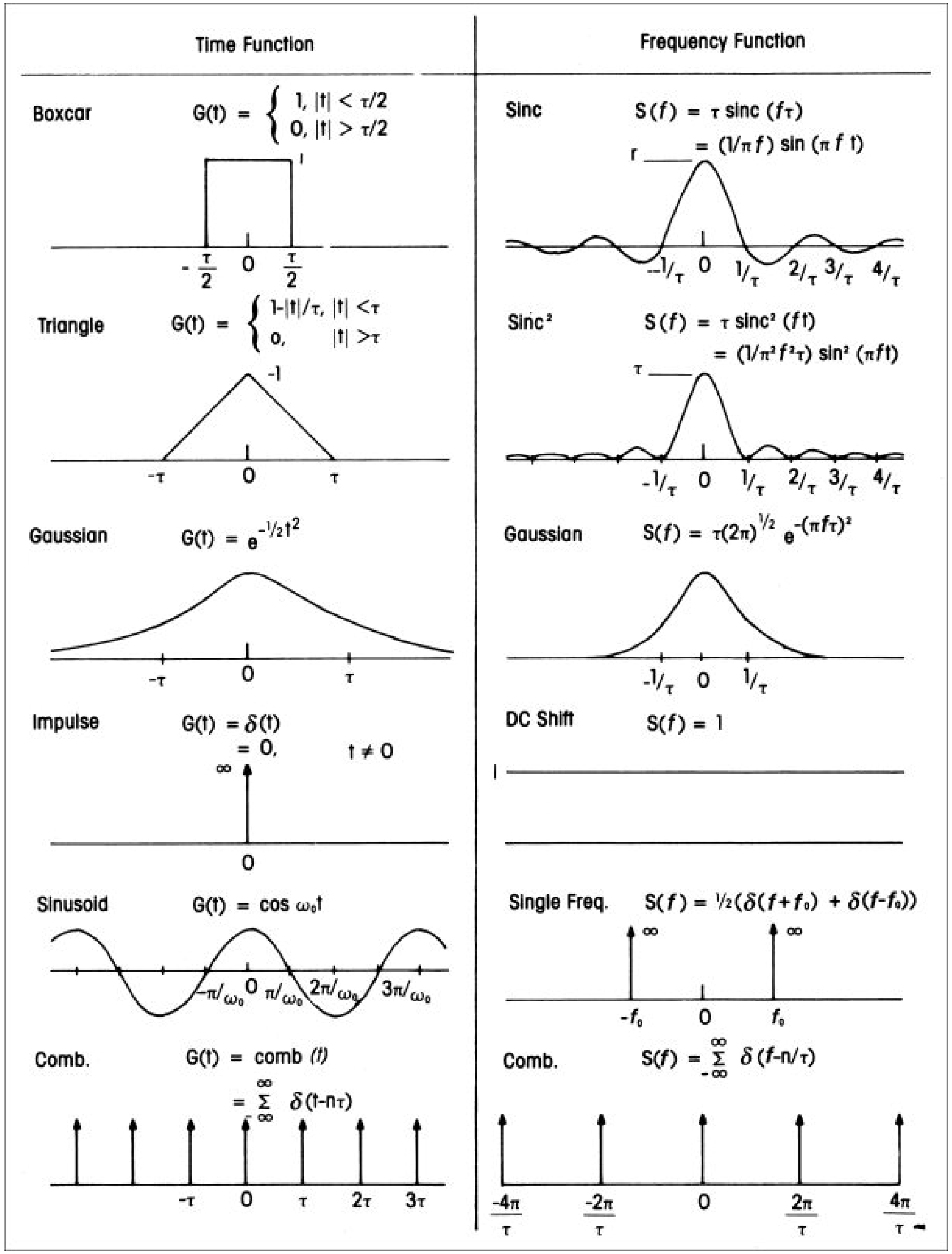
$$F(\omega)=p\frac{sen(\omega p/2)}{\omega p/2}=p\,\mathrm{sinc}(\omega p/2)$$

$$f(t)=\begin{cases}0 & t<\frac{-p}{2}\\ 1 & \frac{-p}{2}<t<\frac{p}{2}\\ 0 & \frac{p}{2}<t\end{cases}$$



En forma gráfica, la transformada es: $F(\omega)=p\,\mathrm{sinc}(\omega p/2)$





Time Function
Frequency Function
Boxcar
G(t) = { 1, |t| < τ/2 ; 0, |t| > τ/2
-τ/2 0 τ/2
Sinc
S(f) = τ sinc (fτ)
= (1/πf) sin (π f t)
-1/τ 0 1/τ 2/τ 3/τ 4/τ
Triangle
G(t) = { 1-|t|/τ, |t| <τ ; 0, |t| >τ
-τ 0 τ
Sinc²
S(f) = τ sinc² (ft)
= (1/π²f²τ) sin² (πft)
-1/τ 0 1/τ 2/τ 3/τ 4/τ
Gaussian
G(t) = e^{-½t²}
-τ 0 τ
Gaussian
S(f) = τ(2π)^{½} e^{-(πfτ)²}
-1/τ 0 1/τ
Impulse
G(t) = δ(t)
= 0, t ≠ 0
∞
0
DC Shift
S(f) = 1
Sinusoid
G(t) = cos ω₀t
-π/ω₀ 0 π/ω₀ 2π/ω₀ 3π/ω₀
Single Freq.
S(f) = ½(δ(f+f₀) + δ(f-f₀))
∞ ∞
-f₀ 0 f₀
Comb.
G(t) = comb (t)
= Σ δ(t-nτ)
-τ 0 τ 2τ 3τ
Comb.
S(f) = Σ δ(f-n/τ)
-4π/τ -2π/τ 0 2π/τ 4π/τ

Fin de la Reseña Matemática

Teorema central del Límite

Si x_i son variables aleatorias independientes con media **a** y varianza **b**^{2}

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i$$

X tendrá una ***distribución gaussiana*** con:

$$E(x) = na \quad , \qquad \sigma^2(x) = nb^2$$

Demostración: $x_i' = x_i - a$, entonces $\varphi'(0) = 0$, $\varphi''(0) = -\sigma^2$

$$\varphi_{x'}(t) = 1 - \frac{1}{2}\sigma^2 t^2 + \cdots$$ Si escribimos $$u_i = \frac{x_i'}{b\sqrt{n}} = \frac{x_i - a}{b\sqrt{n}}$$

$$\varphi_{u_i}(t) = E\{\exp(itu_i)\} = E\left\{\exp\left(it\frac{x_i - a}{b\sqrt{n}}\right)\right\} = \varphi_{x_i'}\left(\frac{t}{b\sqrt{n}}\right)$$

$$\varphi_{u_i}(t) = 1 - \frac{t^2}{2n} + \cdots \quad .$$

Teorema central del Límite

$$\varphi_{u_i}(t) = 1 - \frac{t^2}{2n} + \cdots \quad .$$

$$\mathsf{u} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathsf{u}_i = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mathsf{x}_i - a}{b\sqrt{n}} = \lim_{n\to\infty} \frac{(\mathsf{x} - na)}{b\sqrt{n}}$$

$$\varphi_u(t) = \lim_{n\to\infty} \{\varphi_{u_i}(t)\}^n = \lim_{n\to\infty} \left(1 - \frac{t^2}{2n} + \cdots\right)^n$$

$$\varphi_u(t) = \exp\left(-\frac{1}{2}t^2\right) \qquad E(\mathsf{u}) = 0, \sigma^2(\mathsf{u}) = 1$$

Gaussiana

Teniendo en cuenta que la relación de u con x (variable estandarizada y variable real). Y que esta transformación es lineal

X tendrá una ***distribución gaussiana*** con: $E(\mathsf{x}) = na \quad , \quad \sigma^2(\mathsf{x}) = nb^2$

Función de Distribución Gaussiana Conjunta

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

$$\phi(\mathbf{x}) = k \exp\{-\frac{1}{2}(\mathbf{x}-\mathbf{a})^{\mathrm{T}} B(\mathbf{x}-\mathbf{a})\} = k \exp\{-\frac{1}{2} g(\mathbf{x})\}$$

$$g(\mathbf{x}) = (\mathbf{x}-\mathbf{a})^{\mathrm{T}} B(\mathbf{x}-\mathbf{a}) \quad .$$

Por simetría con respecto al **a**

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\mathbf{x}-\mathbf{a}) \phi(\mathbf{x}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n = 0 \quad ,$$

$$E(\mathbf{x}-\mathbf{a}) = 0$$

$$E(\mathbf{x}) = \mathbf{a} \quad .$$

Función de Distribución Gaussiana Conjunta

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [I - (\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{a})^{\mathrm{T}} B]\phi(\mathbf{x})\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n = 0 \quad .$$

$$\downarrow$$

$$E\{(\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{a})^{\mathrm{T}}\}\,B = I$$

$$C = E\{(\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{a})^{\mathrm{T}}\} = B^{-1}$$

Para el caso de dos variables aleatorias:

$$C = B^{-1} = \begin{pmatrix} \sigma_1^2 & \mathrm{cov}(\mathsf{x}_1, \mathsf{x}_2) \\ \mathrm{cov}(\mathsf{x}_1, \mathsf{x}_2) & \sigma_2^2 \end{pmatrix}$$

$$B = \frac{1}{\sigma_1^2 \sigma_2^2 - \mathrm{cov}(\mathsf{x}_1, \mathsf{x}_2)^2} \begin{pmatrix} \sigma_2^2 & -\mathrm{cov}(\mathsf{x}_1, \mathsf{x}_2) \\ -\mathrm{cov}(\mathsf{x}_1, \mathsf{x}_2) & \sigma_1^2 \end{pmatrix}$$

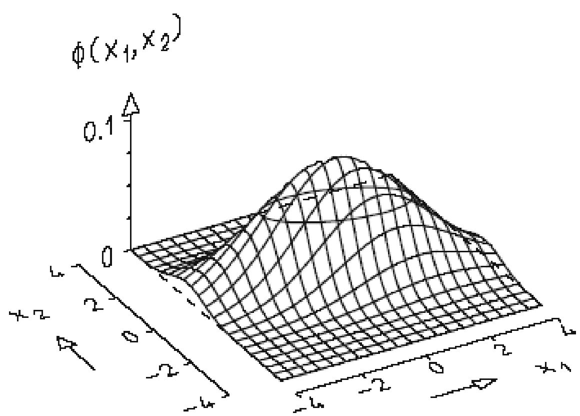
Función de Distribución Gaussiana Conjunta

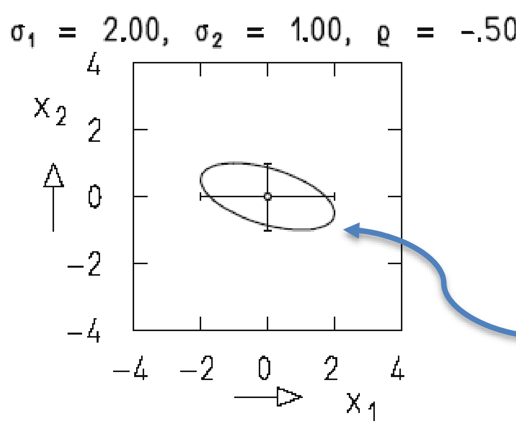
$$\phi(\mathbf{x}) = k \exp\{-\frac{1}{2}(\mathbf{x}-\mathbf{a})^{\mathrm{T}} B(\mathbf{x}-\mathbf{a})\} \qquad k = \left(\frac{\det B}{(2\pi)^n}\right)^{\frac{1}{2}}$$

Comentario referente a si las dos variables son independientes……

Dibujemos la gaussiana conjunta:

Análisis de la matriz de varianza covarianza….





$z = \phi(x_1, x_2) = ec.\, gaussiana$

$z = Cte$

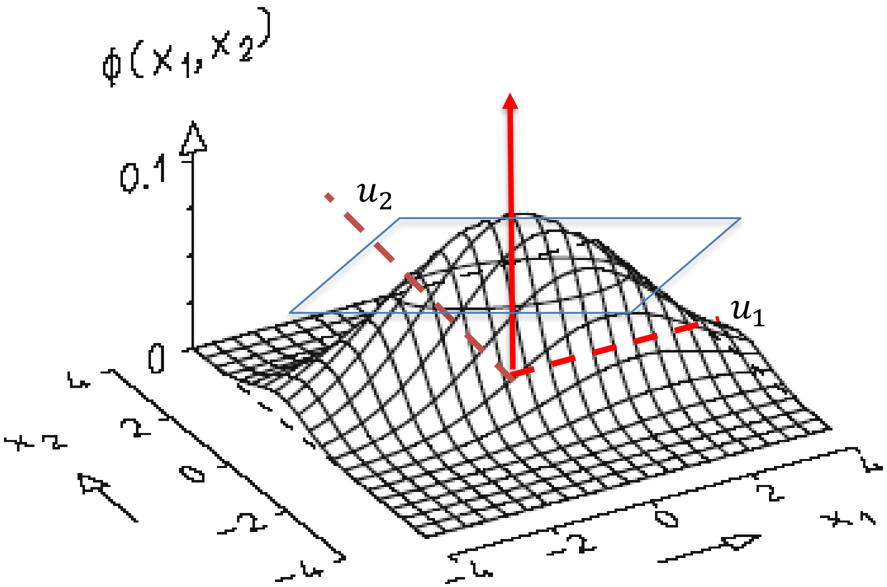
Finalmente:

$\phi(x_1, x_2)$=Cte

Función de Distribución Gaussiana Conjunta

$$\mathrm{u}_i = \frac{\mathrm{x}_i - a_i}{\sigma_i} \quad , \qquad i = 1, 2 \quad ,$$

$$\rho = \frac{\mathrm{cov}(\mathrm{x}_1, \mathrm{x}_2)}{\sigma_1 \sigma_2} = \mathrm{cov}(\mathrm{u}_1, \mathrm{u}_2)$$



$$= k \exp(-\frac{1}{2}\mathbf{u}^{\mathrm{T}} B \mathbf{u}) = k \exp\left(-\frac{1}{2} g(\mathbf{u})\right)$$

$$B = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \quad .$$

$$-\frac{1}{2}\cdot\frac{1}{(1-\rho^2)}(u_1^2+u_2^2-2u_1u_2\rho) = -\frac{1}{2}g(\mathbf{u}) = \text{const}$$

$$g(\mathbf{u}) = 1.$$

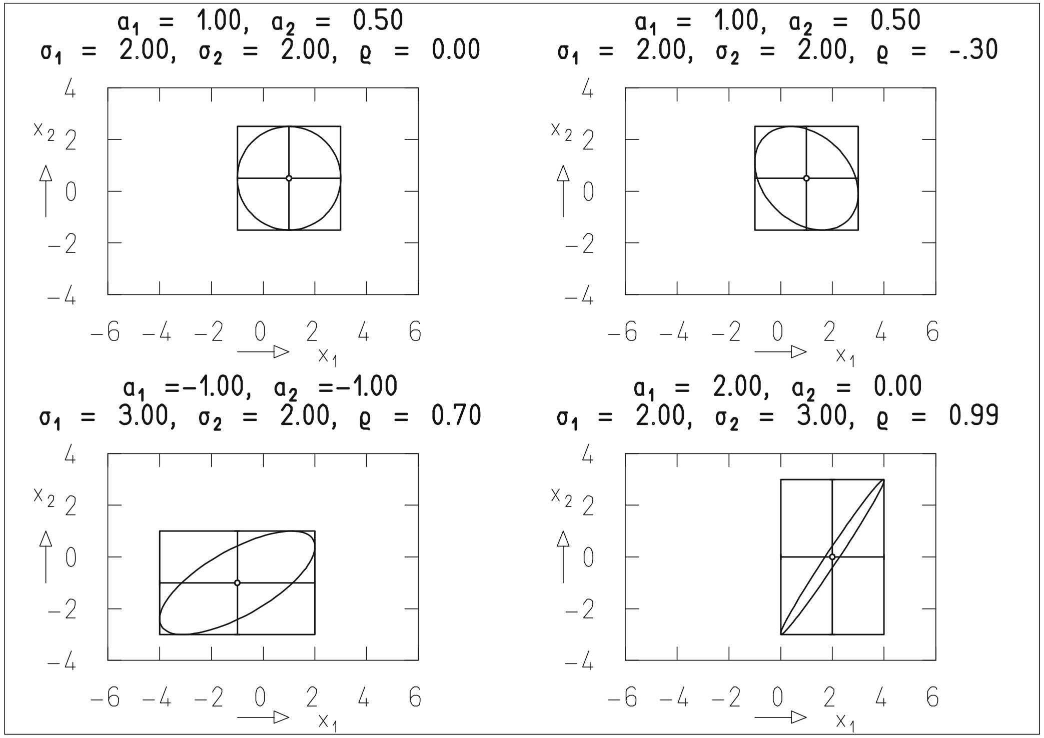
$$\frac{(x_1-a_1)^2}{\sigma_1^2} - 2\rho\frac{x_1-a_1}{\sigma_1}\frac{x_2-a_2}{\sigma_2} + \frac{(x_2-a_2)^2}{\sigma_2^2} = 1-\rho^2$$

Elipse de covarianza

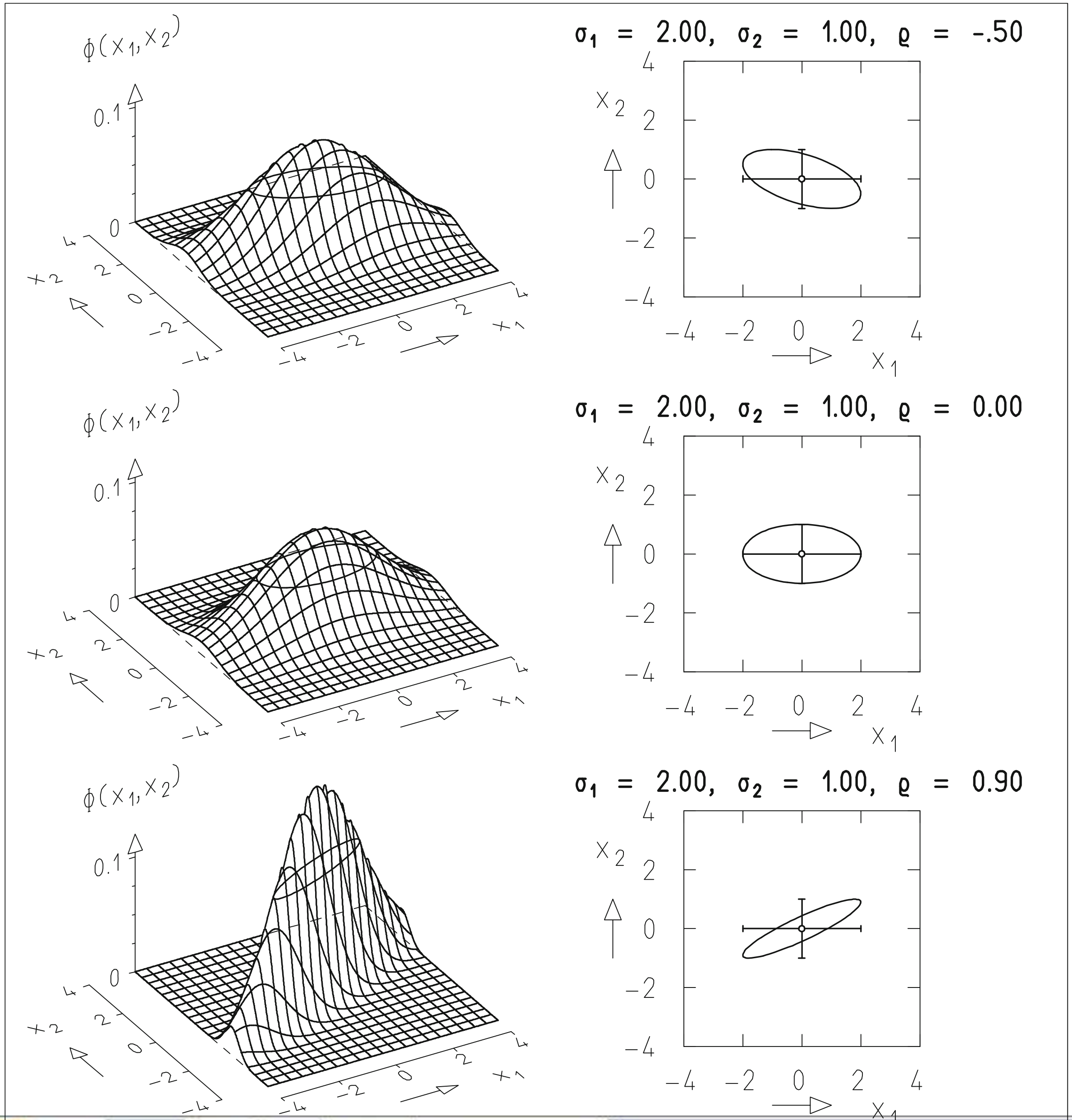
$$\tan 2\alpha = \frac{2\rho\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2} \quad ,$$

$$p_1^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_2^2\cos^2\alpha - 2\rho\sigma_1\sigma_2\sin\alpha\cos\alpha + \sigma_1^2\sin^2\alpha} \quad ,$$

$$p_2^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_2^2\sin^2\alpha + 2\rho\sigma_1\sigma_2\sin\alpha\cos\alpha + \sigma_1^2\cos^2\alpha} \quad .$$



a1 = 1.00, a2 = 0.50
σ1 = 2.00, σ2 = 2.00, ρ = 0.00
a1 = 1.00, a2 = 0.50
σ1 = 2.00, σ2 = 2.00, ρ = -.30
a1 =−1.00, a2 =−1.00
σ1 = 3.00, σ2 = 2.00, ρ = 0.70
a1 = 2.00, a2 = 0.00
σ1 = 2.00, σ2 = 3.00, ρ = 0.99
x1
x2
4
2
0
−2
−4
−6
−4
−2
0
2
4
6



σ1 = 2.00, σ2 = 1.00, ϱ = -.50
σ1 = 2.00, σ2 = 1.00, ϱ = 0.00
σ1 = 2.00, σ2 = 1.00, ϱ = 0.90
ϕ(x1,x2)
0.1
0
x1
x2
X1
X2