Función de distribución Normal (o Gaussiana) conjunta

Función Normal o de Gauss para una variable:

$$X \longrightarrow f(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma_X} \exp\left(-\frac{1}{2}\,\frac{(x-\mu_X)^2}{\sigma_X^2}\right)$$

$$= \frac{1}{\sqrt{2\,\pi}\,\sigma_X} \exp\left(-\frac{1}{2}\,(x-\mu_X)\frac{1}{\sigma_X^2}(x-\mu_X)\right)$$

Generalización para N variables:

$$\overline{X} = (X_1, X_2, \cdots, X_N)^T \longrightarrow \Phi(\overline{x}) = K \exp\left(-\frac{1}{2}\,(\overline{x}-\overline{a})^t\, B\,(\overline{x}-\overline{a})\right)$$

$$\Phi(\overline{X}) \colon \mathbb{R}^N \to \mathbb{R}$$

K ⟶ Factor de normalización

$\overline{a}$ ⟶ Vector de N x 1

B ⟶ Matriz de N x N, simétrica y definida positiva

Características de Función de distribución Normal conjunta

1. El valor esperado de $\bar{X}$ es: $E[\bar{X}] = \bar{a}$

$\Phi(\bar{X})$ es simétrica respecto de $\bar{a}$, luego:

$$\iiint_{-\infty}^{+\infty} (\overline{x} - \overline{a})\, \Phi(\overline{x})\, \mathrm{d}\overline{x} = 0$$

$$E[(\overline{x} - \overline{a})] = 0 \longrightarrow E[\overline{x}] = \overline{a}$$

$\therefore$ $\bar{a}$ es la media de $\bar{x}$ por lo que se lo escribe como $\bar{a} = \bar{\mu}$

2. Se cumple que: $B = C_{\bar{x}}^{-1}$

Derivando la integral de punto 1 respecto de $\bar{a}$

$$\frac{\partial}{\partial \bar{a}}\left(\iiint_{-\infty}^{+\infty} (\bar{x}-\bar{a})\, \Phi(\bar{x})\, d\bar{x} \right) = 0$$

$$\frac{\partial}{\partial \bar{a}}\left(\iiint_{-\infty}^{+\infty} (\bar{x}-\bar{a})\ K\, e^{-\frac{1}{2}(\bar{x}-\bar{a})^t B\, (\bar{x}-\bar{a})}\, d\bar{x} \right) = 0$$

$\frac{\partial}{\partial \bar{a}}\left((\bar{x}-\bar{a})^t\, B\, (\bar{x}-\bar{a})\right) = -2(\bar{x}-\bar{a})^t B$, para B símétrica

$$\int_{-\infty}^{+\infty} \left[-IK\, e^{-\frac{1}{2}(\bar{x}-\bar{a})^t B\, (\bar{x}-\bar{a})} + (\bar{x}-\bar{a})\ \frac{-2}{-2}(\bar{x}-\bar{a})^t\, B\, K e^{-\frac{1}{2}(\bar{x}-\bar{a})^t B\, (\bar{x}-\bar{a})}\right] \mathrm{d}\bar{x} = 0$$

$$\int_{-\infty}^{+\infty} [-I + (\bar{x}-\bar{a})\ (\bar{x}-\bar{a})^t\, B]\, \Phi(\bar{x})\, \mathrm{d}\bar{x} = 0$$

$$E[(\bar{x}-\bar{a})\ (\bar{x}-\bar{a})^t]\, B = I \int_{-\infty}^{+\infty} \Phi(\bar{x})\, \mathrm{d}\bar{x}$$

$$C_{\bar{X}}\, B = I$$

$$B = C_{\bar{X}}^{-1}$$

$\therefore$ B es la inversa de la matriz de var-covar de $\bar{x}$

Características de la Función de distribución Normal de dos variables (Función normal bi-variada)

$$\overline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad \Phi(\overline{x}) = K \exp\left(-\frac{1}{2}\,(\overline{x}-\overline{\mu})^t\, B\,(\overline{x}-\overline{\mu})\right)$$

$$\Phi\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K \exp\left(-\frac{1}{2}\begin{bmatrix} x_1 - \mu_{X_1} \\ x_2 - \mu_{X_2} \end{bmatrix}^t B \begin{bmatrix} x_1 - \mu_{X_1} \\ x_2 - \mu_{X_2} \end{bmatrix}\right)$$

Donde:

$$B = C^{-1} = \begin{bmatrix} \sigma_{X_1}^2 & cov(X_2, X_1) \\ cov(X_1, X_2) & \sigma_{X_2}^2 \end{bmatrix}^{-1}$$

$$B = \frac{1}{\sigma_{X_1}^2\sigma_{X_2}^2 - cov(X_1, X_2)^2}\begin{bmatrix} \sigma_{X_2}^2 & -cov(X_2, X_1) \\ -cov(X_1, X_2) & \sigma_{X_1}^2 \end{bmatrix}$$

$$K = \frac{1}{(2\,\pi)(\det B)^{1/2}}$$ → Factor de normalización

1. Distribución normal de dos variables con $cov(X1, X2) = 0$

$$B = \frac{1}{\sigma_{X_1}^2 \sigma_{X_2}^2 - cov(X_1, X_2)^2} \begin{bmatrix} \sigma_{X_2}^2 & -cov(X_2, X_1) \\ -cov(X_1, X_2) & \sigma_{X_1}^2 \end{bmatrix} \longrightarrow B = \begin{bmatrix} \frac{1}{\sigma_{X_1}^2} & 0 \\ 0 & \frac{1}{\sigma_{X_2}^2} \end{bmatrix}$$

$$K = \frac{1}{(2\,\pi)\,(\det B)^{1/2}} \longrightarrow K = \frac{1}{(2\,\pi)\,\sigma_{X_1}\sigma_{X_2}}$$

$$\therefore$$

$$\Phi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{(2\,\pi)\,\sigma_{X_1}\sigma_{X_2}} \exp\left[-\frac{1}{2}\left(\left(\frac{x_1 - \mu_{X_1}}{\sigma_{X_1}}\right)^2 + \left(\frac{x_2 - \mu_{X_2}}{\sigma_{X_2}}\right)^2\right)\right]$$

Podemos verificar que la probabilidad total es 1:

$$\iint_{-\infty}^{+\infty} \Phi(x_1, x_2)\, dx_1 dx_2 = \iint_{-\infty}^{+\infty} \frac{1}{(2\,\pi)\,\sigma_{X_1}\sigma_{X_2}} \exp\left[-\frac{1}{2}\left(\left(\frac{x_1-\mu_{X1}}{\sigma_{X_1}}\right)^2 + \left(\frac{x_2-\mu_{X2}}{\sigma_{X_2}}\right)^2\right)\right] dx_1 dx_2$$

$$= \frac{1}{(2\,\pi)\,\sigma_{X_1}\sigma_{X_2}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x_1-\mu_{X1}}{\sigma_{X_1}}\right)^2\right) dx1 \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x_2-\mu_{X2}}{\sigma_{X_2}}\right)^2\right) dx2$$

$$\iint_{-\infty}^{+\infty} \Phi(x_1, x_2)\, dx_1 dx_2 = \frac{1}{(2\,\pi)\,\sigma_{X_1}\sigma_{X_2}} \sqrt{2\pi}\sigma_{X_1}\sqrt{2\pi}\sigma_{X_2}$$

$$\iint_{-\infty}^{+\infty} \Phi(x_1, x_2)\, dx_1 dx_2 = 1$$

2. Distribución normal de dos variables con $cov(X1, X2) \neq 0$

$$\Phi\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K\,\exp\left(-\frac{1}{2}\begin{bmatrix} x_1-\mu_{X\,1} \\ x_2-\mu_{X\,2} \end{bmatrix}^t B\begin{bmatrix} x_1-\mu_{X\,1} \\ x_2-\mu_{X\,2} \end{bmatrix}\right)$$

Donde:

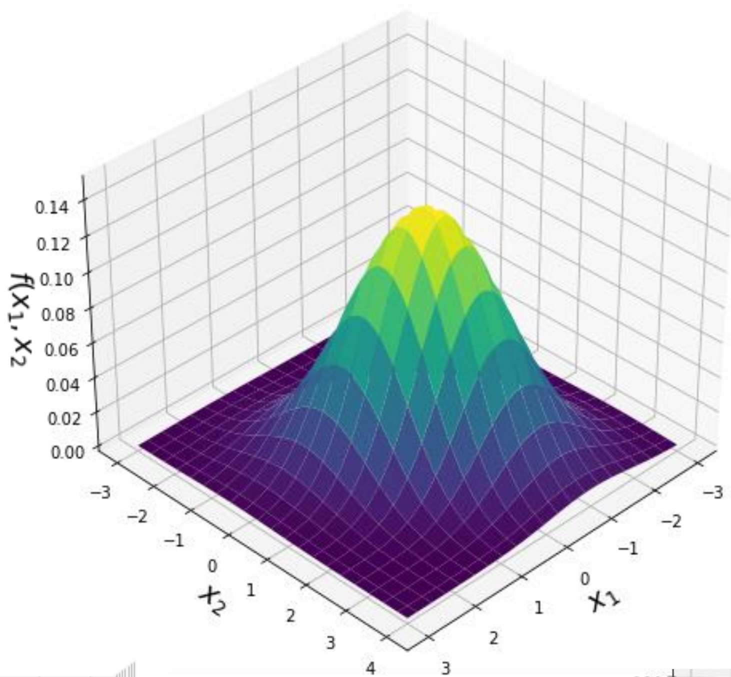
$$K = \frac{1}{(2\,\pi)\,(\det B)^{1/2}}$$

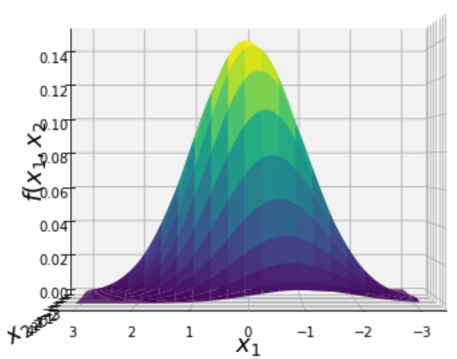
$$B = C_{X_1,X_2}{}^{-1} \qquad C_{X_1,X_2} = \begin{bmatrix} \sigma_{X_1}^2 & cov(X_1,X_2) \\ cov(X_2,X_1) & \sigma_{X_2}^2 \end{bmatrix}^{-1}$$

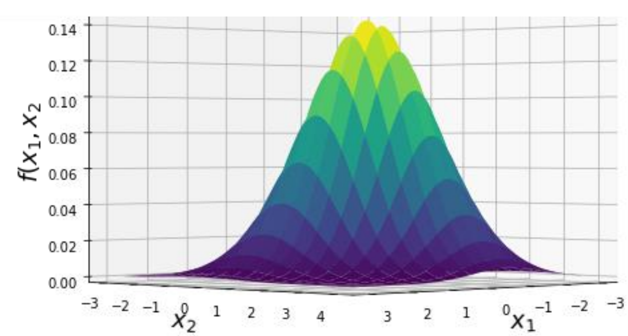
$$B = \frac{1}{\sigma_{X_1}^2\sigma_{X_2}^2 - cov(X_2,X_1)^2}\begin{bmatrix} \sigma_{X_2}^2 & -cov(X_2,X_1) \\ -cov(X_1,X_2) & \sigma_{X_1}^2 \end{bmatrix}$$

Representación gráfica:

$$\Phi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K \exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_{X_1} \\ x_2 - \mu_{X_2} \end{bmatrix}^t B \begin{bmatrix} x_1 - \mu_{X_1} \\ x_2 - \mu_{X_2} \end{bmatrix} \right)$$







Análisis de la matriz de varianza – covarianza

Transformamos las variables según:

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, i = 1,2 \longrightarrow cov(U_1, U_2) = \frac{cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} \equiv \rho_{X_1,X_2}$$

ρ_{X_1,X_2} Se llama coeficiente de correlación

Luego, la densidad gaussiana conjunta queda expresada como: :

$$\Phi \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = K \exp\left(-\frac{1}{2} g(u_1, u_2)\right)$$

donde:

$$g(u_1, u_2) = \bar{u}^t \, B' \, \bar{u} \qquad B' = \frac{1}{1 - \rho_{X_1,X_2}} \begin{bmatrix} 1 & -\rho_{X_1,X_2} \\ -\rho_{X_1,X_2} & 1 \end{bmatrix}$$

$$g(u_1, u_2) = \frac{1}{1 - \rho_{X_1,X_2}} \left(u_1^2 + u_2^2 - 2\rho_{X_1,X_2} u_1 u_2 \right)$$

Elipse de covarianza

Las curvas de igual probabilidad quedan definidas para:

$$\Phi\begin{bmatrix}u_1\\u_2\end{bmatrix} = Const1 \longrightarrow g(u_1, u_2) = Const2$$

En particular, cuando $Const2 = 1$:

$$\frac{1}{1-\rho_{X_1,X_2}}\left(u_1^2 + u_2^2 - 2\rho_{X_1,X_2}u_1u_2\right) = 1$$

Expresando la ecuación en las variables originales y reacomodando:

$$\frac{\left(x_1-\mu_{X_1}\right)^2}{\sigma_{x_1}^2} - 2\rho_{X_1,X_2}\left(\frac{x_1-\mu_{X_1}}{\sigma_{x_1}}\right)\left(\frac{x_1-\mu_{X_2}}{\sigma_{x_2}}\right) + \frac{\left(x_2-\mu_{X_2}\right)^2}{\sigma_{x_2}^2} = 1-\rho_{X_1,X_2}$$

La anterior ecuación define la elipse de covarianza y corresponde a la curva tal que:

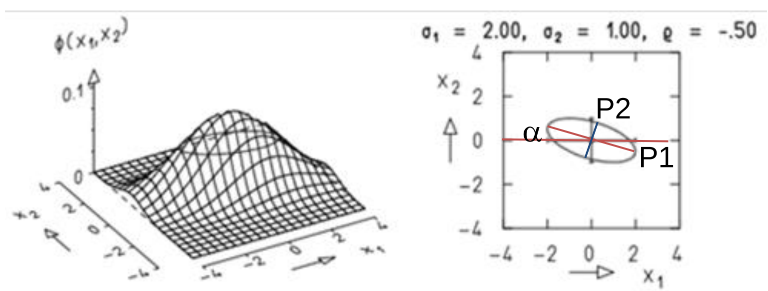
$$\iint_{Area\ de\ la\ elipse} \Phi(x_1, x_2)\, dx_1 dx_2 \approx 0.39$$

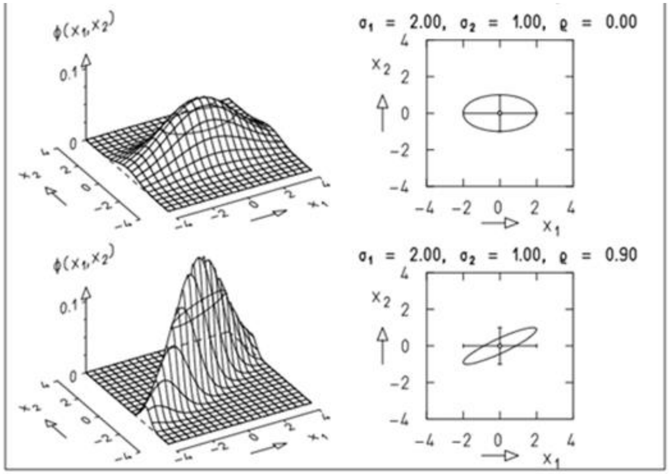
Los parámetros de la elipse se puede determinar a partir de:

Angulo → $$\tan 2\alpha = \frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2} \quad ,$$

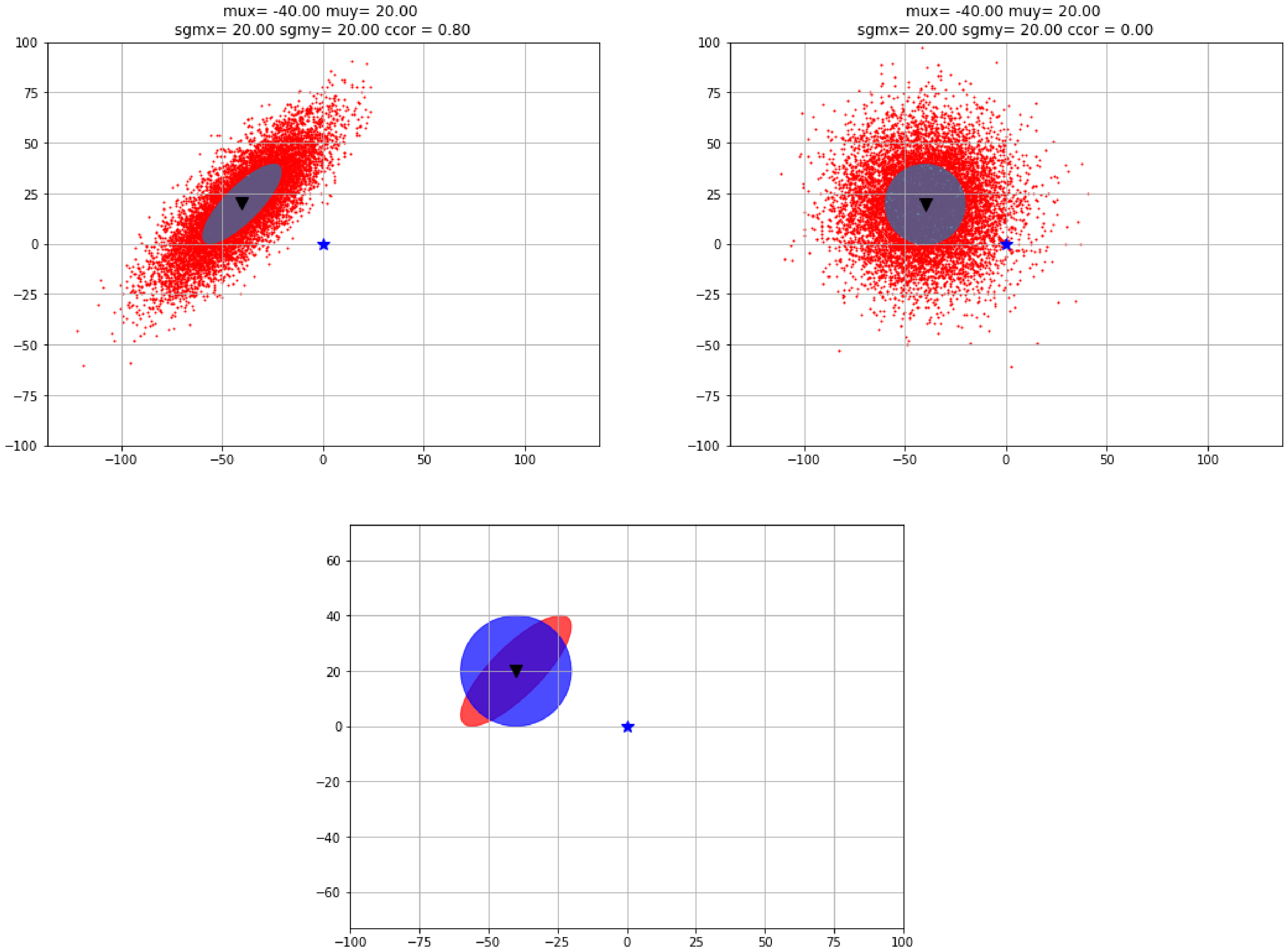
Semi-eje mayor → $$p_1^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_2^2\cos^2\alpha - 2\rho\sigma_1\sigma_2\sin\alpha\cos\alpha + \sigma_1^2\sin^2\alpha}$$

Semi-eje menor → $$p_2^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_2^2\sin^2\alpha + 2\rho\sigma_1\sigma_2\sin\alpha\cos\alpha + \sigma_1^2\cos^2\alpha}$$





Experimento función normal conjunta con JupyterLab (Gaussiana1D.Ipynb, Gaussian2D – Surface.ipynb y Gaussiana2D – Ellipses.ipynb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